Ergodic Theory - Week 6

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1 Classifying measure preserving systems

P1. Prove that the systems $\mathbb{X} = (\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}}, R_\alpha)$ and $\mathbb{Y} = (\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}}, T_2)$ are not isomorphic, where $R_\alpha x = x + \alpha \pmod 1$ and $T_2 x = 2x \pmod 1$.

P2. Let $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq \mathbb{T}$. Show that there is an increasing sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that for every $k \in \mathbb{N}$,

$$||n_k \alpha_l||_{\mathbb{T}} \leq \frac{1}{k}, \quad \forall l \in \{1, \ldots, k\}.$$

Hint: Use Poincaré's Theorem in a convenient group rotation.

P3. Prove that the systems $\mathbb{X} = (\mathbb{T}, \mathcal{B}(\mathbb{T}), m_{\mathbb{T}}, T_4)$ and $\mathbb{Y} = (\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), m_{\mathbb{T}^2}, T_2 \times T_2)$ are isomorphic.

P4. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure-preserving system and let A be a set of positive measure. The pointwise ergodic theorem shows that for almost all $x \in X$, the set of visiting times

$$\Lambda_x = \{n \in \mathbb{N} \colon T^n x \in A\}$$

has natural density equal to $\mu(A)$. Is it true that, for almost all $x \in X$, the set Λ_x has bounded gaps?